COMPLEX PRODUCT STRUCTURES ON SOME SIMPLE LIE GROUPS

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Abstract. We construct invariant complex product (hyperparacomplex, indefinite quaternion) structures on the manifolds underlying the real noncompact simple Lie groups $SL(2m-1,\mathbb{R})$, $SU(m,m-1)$ and $SL(2m-1,\mathbb{C})^{\mathbb{R}}$. We show that on the last two series of groups some of these structures are compatible with the biinvariant Killing metric. Thus we also provide a class of examples of compact (neutral) hyperparahermitean, non-flat Einstein manifolds.

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1. Introduction

In the present paper we construct explicit examples of homogeneous complex product structures on real semisimple Lie groups and some related manifolds. Our examples fall into the class of CPS known also as hyperparacomplex structures. The relevant definitions are provided in Section 2.

The main result is

Theorem 1.1. *For each $m > 1$, the manifolds underlying the Lie groups $SL(2m-1,\mathbb{R})$ and $SU(m,m-1)$ have complex product structures, which are invariant by left translations.*

The proof of the above theorem will be given in Section 3, there we construct explicitely a family of complex product structures depending on parameters in each of the above Lie groups. In this introduction we indicate some more examples of complex product manifolds, obtained directly from Theorem 1.1, and discuss properties of their complex product structures. For the sake of brevity, we sometimes speak of complex product structures on Lie algebras, rather than on the manifolds underlying the respective Lie groups. Reduction of the latter to the former is explained in Section 2 (see also e.g. [1]).

To our knowledge (see also Andrada and Salamon [1] p.2) Theorem 1.1 provides the first examlpes of homogeneous complex product structures on semisimple real Lie groups. There are many known complex product structures on solvable groups. Andrada and Salamon [1] have constructed invariant complex product structures on the reductive groups $GL(2m,\mathbb{R})$. From the proof of Theorem 1.1, we shall obtain complex product structures on the groups $GL(2m,\mathbb{R})$, embedding[1] them as (complex product) submanifolds of $SL(2m+1,\mathbb{R})$ (i.e. inheriting the complex product structure of Theorem 1.1). In the same way we obtain complex product structures on $U(m,m)$.

Let $\mathfrak{g}$ be a real Lie algebra, and let (P,J) be a complex product structure on $\mathfrak{g}$. It is trivial to check, that if we extend P,J to the complexification $\mathfrak{g}^{\mathbb{C}}$ by linearity ($P(iX) \doteq iP(X)$, $J(iX) \doteq iJ(X)$), then we get a complex product structure on the real Lie algebra $\left(\mathfrak{g}^{\mathbb{C}}\right)^{\mathbb{R}}$. Thus we have

Corollary 1.2. *For each $m > 1$, the manifold underlying $(SL(2m-1,\mathbb{C}))^{\mathbb{R}}$ admits homogeneous complex product structures.*

To get a more interesting and subtle result we combine Theorem 1.1 with Theorem 3.3 of Andrada and Salamon [1] where they prove, that a complex product structure on a real Lie

[1] see Remark 3.2.

algebra $\mathfrak{g}$ induces a hypercomplex structure on $\mathfrak{g}^{\mathbb{C}}$. As $sl(2m-1,\mathbb{C}) = (sl(2m-1,\mathbb{R}))^{\mathbb{C}} = ((su(m,m-1))^{\mathbb{C}})$ we get

Corollary 1.3. *For each $m > 1$, the manifold underlying $SL(2m-1,\mathbb{C})$ admits homogeneous hypercomplex structures.*

Let $\mathfrak{g}$ be a real Lie algebra with a complex product structure (P, J). Each nondegenerate symmetric bilinear form $g(X, Y)$ on $\mathfrak{g}$, satisfying the condition (4), defines an invariant pseudoriemannian metric (neutral) on the manifold G (the simply connected group whose Lie algebra is $\mathfrak{g}$), which is compatible[2] with (P, J). Thus the homogeneous complex product structures of Theorem 1.1 and Corollary 1.2 admit compatible left invariant metrics. On $SL(2m-1,\mathbb{R})$ the Killng form is not of neutral signature, so it cannot be compatible with any complex product structure. On the other hand we have

Theorem 1.4. *For each $m > 1$ the group manifolds $SU(m, m-1)$ and $(Sl(2m-1,\mathbb{C}))^{\mathbb{R}}$ admit complex product structures, which are compatible with the biinvariant metric induced by the respective Killing forms.*

The proof will also be given in Section 3.

Let (P, J) be a left invariant complex product structure on a Lie group G. If $\Gamma \subset G$ is a discrete subgroup, and Γ acts on G by left translations, then, obviously, (P,J) descends to a complex product structure on the factor G/Γ.

Further by a famous theorem of A. Borel [4], each connected semisimple Lie group G admits uniform discrete subgroups. The Killing metric (of any signature) is Einstein (see e.g. [2]). Thus by Theorem 1.4 the interesting class of examples provided by the next Corollary is not void.

Corollary 1.5. *Let (P, J) be an invariant complex product structure compatible with the Killing form on $SU(m, m-1)$ (respectively $(Sl(2m-1,\mathbb{C}))^{\mathbb{R}}$ or any simple Lie group). The factors $SU(m, m-1)/\Gamma$ by any cocompact discrete subgroup Γ are compact complex product manifolds with compatible non-flat neutral Einstein metric.*

We should remark that non-flat compact complex product manifolds (See e.g. [7] and references there) have been known for some time.

The complex product manifolds introduced in the present paper, are treated in the context of para quaternionic differential geometry in [6].

2. PRELIMINARIES

The definitions and standard notation in Lie group and Lie algebra theory used in this paper can be found e.g. in [5]. For completenes we review briefly the definitions and standard facts arround the notion of complex product structure. For a comprehensive introduction to the subject we recommend [1].

Deffinition 2.1. *Let M be a manifold and let TM be the tangent bundle of M. An* **almost product structure** *on M is a fibrewise linear involution, $P : TM \longmapsto TM$ (of constant trace).*

An integrable almost product structure on M will be called **product structure**.

A **complex product structure** *on M is a couple (P, J), where P is a product structure, J is a complex structure, and we have*

$$PJ = -JP. \tag{1}$$

[2]See Definition 2.2.

Recall that integrability of P, respectively J, by definition, means that

$$
\begin{aligned}
N_P(X,Y) &\doteq [P(X),P(Y)] + [X,Y] - P([P(X),Y]) - P([X,P(Y)]) = 0;\\
N_J(X,Y) &\doteq [J(X),J(Y)] - [X,Y] - J([J(X),Y]) - J([X,J(Y)]) = 0,
\end{aligned}
\tag{2}
$$

for any choice of vector fields X, Y on M.

It is easy to see that if P is the product structure in a complex product structure (P, J), then the condition (1) implies $trP = 0$. This means that if we decompose the tangent space TM at a point $m \in M$, into eigenspaces

$$P^+ = P^+_m \doteq \{X \in T_m : PX = X\}, \quad P^- = P^-_m \doteq \{X \in T_m : PX = -X\},$$

then $dimP^+ = dimP^-$ at each point $m \in M$.

Remark 2.1. *Product structures P with zero trace were called* **para-complex** *structures by P. Liebermann in the early fifties. They have been studied by many authors. If (P, J) is a complex product structure on M and we denote $Q = JP$, then (Q, J) is another complex product structure on M. It is known (in the general case), that the integrability of Q is a consequence of (P, J) being a complex product structure ([7],[3]). Obviously the situation is symmetric, we have three anticommuting operators P, Q, J with*

$$
\begin{aligned}
&P^2 = Q^2 = -J^2 = 1,\\
JP = -PJ = Q, \quad &QP = -PQ = J, \quad QJ = -JQ = P.
\end{aligned}
\tag{3}
$$

The Lie algebra generated by P, Q, J is isomorphic to $sl(2, \mathbb{R}) \cong su(1, 1)$. The examples constructed in Section 3 are such, that the tangent spaces decompose into real four[3] dimmensional irreducible representations of $sl(2, \mathbb{R})$, all of them coinciding with the standard representation of $su(1, 1)$ in $\mathbb{C}^2$. In the literature such complex product manifolds are sometimes called **hyperparacomplex**[4]. *There is a natural class of pseudoriemannian metrics on such a manifold, which we proceed to define.*

Deffinition 2.2. *We shall say that a pseudoriemannian metric g and a complex product structure (P, J) on a manifold M are* **compatible**[5] *if*

$$g(J(X), J(Y)) = g(X, Y), \quad g(P(X), P(Y)) = -g(X, Y), \tag{4}$$

for arbitrary tangent vectors X, Y at a point of M.

An easy check shows, that a compatible metric has neutral signature, and that P^+, P^- have to be isotropic subspaces. A study of such metrics and related connections can be found in [6].

Let G be a simply connected real Lie group of finite dimension, and let $\mathfrak{g}$ be the Lie algebra of G, which we identify with the space of left invariant vector fields on G, thus trivializing the tangent bundle of G. It is obvious, that a couple of linear operators $P, J : \mathfrak{g} \longmapsto \mathfrak{g}$, such that

$$P^2 = 1, \quad J^2 = -1, \quad PJ = -JP, \tag{5}$$

defines a complex product structure on G if and only if the Nijenhuis conditions (2) hold for each $X, Y \in \mathfrak{g}$.

All left translations with elements of the group G acting on itself are in this case holomorphic transformations of the manifold G (with respect to the complex structure J), and also preserve

[3]The standard real two dimmensional representation of $sl(2, \mathbb{R})$ also appears in some examples of CP manifolds, we do not treat this case here.

[4]In algebra and the theory of arithmetic subgroups, the algebra over $\mathbb{R}$ generated by $1, J, P, Q$ with (3) satisfied is known as the **indefinite quaternions**.

[5]It is also sensible to consider metrics such that $g(P(X), P(Y)) = g(X, Y)$, instead of the condition (4). We do not treat this case here.

the product structure P. We say in this case that the complex product structure (P, J) is **(left) invariant** or more loosely that G is a **homogeneous complex product** space[6].

It is well known that right translations on a Lie group G are also holomorphic with respect to a left invariant complex structure J if and only if

$$[J(X), Y] = J([X, Y]), \quad X, Y \in \mathfrak{g}, \tag{6}$$

in this case G is a complex Lie group w.r. to J. It is also known, (Proposition 2.6 in [1]) that only Abelian Lie algebras admit complex product structures where J satisfies (6). So we cannot hope to fall in this case with our simple groups.

In order to make our examlples more transparent, we shall remind here the alternative definition of integrability. Denote

$$P^+ \doteq \{X \in \mathfrak{g} : P(X) = X\}, \quad P^- \doteq \{X \in \mathfrak{g} : P(X) = -X\}$$
$$J^+ = \mathfrak{g}^{1,0} \doteq \{X \in \mathfrak{g}^{\mathbb{C}} : J(X) = iX\}, \quad J^- = \mathfrak{g}^{0,1} \doteq \{X \in \mathfrak{g}^{\mathbb{C}} : J(X) = -iX\}.$$

Integrability of (P, J) is equivalent to the condition that $P^+, P^- \quad (J^+, J^-)$ are Lie subalgebras of $\mathfrak{g} \quad (\mathfrak{g}^{\mathbb{C}})$. Anticommutation of P, J means $J(P^+) = P^-$ (see [1]).

3. The complex product structures

Proof. of Theorem 1.1

As usual, we denote by $E_j^k \in gl(n)$ the matrix with entry 1 at the intersection of the j-th row and the k-th column and 0 elsewhere. We shall now define two linear operators (P, J) for a real vector space $\mathfrak{g}$ of dimension $4m^2 - 4m$ with a base consisting of elements

$$U^j, V^j, S^j, T^j, U_j^k, V_j^k, S_j^k, T_j^k,$$

where the range of indices is

$$j = 1, \ldots, m-1, \qquad j < k < 2m - j. \tag{7}$$

The operators (P, J) are defined for each choice of j, k by the conditions:

$$P^2 = 1, \qquad J^2 = -1, \tag{8}$$

and an explicit formula for each j, k:

$$\begin{aligned} &J(U^j) \doteq V^j; \quad J(S^j) \doteq T^j; &\qquad &P(U^j) \doteq T^j, \quad P(V^j) \doteq S^j; \\ &J(U_j^k) \doteq V_j^k; \quad J(S_j^k) \doteq T_j^k; &\qquad &P(U_j^k) \doteq T_j^k; \quad P(V_j^k) \doteq S_j^k. \end{aligned} \tag{9}$$

We introduce respective bases over $\mathbb{R}$ for $sl(2m-1, \mathbb{R})$ and $su(m, m-1)$ as real subalgebras of $gl(2m-1, \mathbb{C})$.

*We claim that the operators (P, J) defined by formulae (8), (9) and applied to the following base of $sl(2m-1, \mathbb{R})$ define a complex product structure:

$$\begin{aligned} U^j &\doteq E_j^j + E_{2m-j}^{2m-j} - 2E_m^m; &\qquad T^j &\doteq E_j^j - E_{2m-j}^{2m-j}; \\ V^j &\doteq E_j^{2m-j} - E_{2m-j}^j; &\qquad S^j &\doteq E_j^{2m-j} + E_{2m-j}^j; \\ U_j^k &\doteq E_j^k - E_k^j; &\qquad V_j^k &\doteq E_k^{2m-j} - E_{2m-j}^k; \\ S_j^k &\doteq E_k^{2m-j} + E_{2m-j}^k; \quad , &\qquad T_j^k &\doteq E_j^k + E_k^j. \end{aligned} \tag{10}$$

**We claim that the operators (P, J) defined by formulae (8), (9) and applied to the following base of $su(m, m-1)$ define a complex product structure:

[6] Obviously the notion of homogeneous (J, P) space merits a wider definition, we shall not need it.

$$
\begin{aligned}
U^j &\doteq i(E^j_j + E^{2m-j}_{2m-j} - 2E^m_m); & V^j &\doteq i(E^j_j - E^{2m-j}_{2m-j});\\
T^j &\doteq i(E^{2m-j}_j - E^j_{2m-j}); & S^j &\doteq E^{2m-j}_j + E^j_{2m-j}.
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
U^k_j &\doteq \begin{cases} E^k_j - E^j_k & \text{if} \quad j < k \leq m;\\ E^{2m-j}_k - E^k_{2m-j} & \text{if} \quad m < k < 2m-j.\end{cases}\\
V^k_j &\doteq \begin{cases} i(E^k_j + E^j_k) & \text{if} \quad j < k \leq m;\\ i(E^{2m-j}_k + E^k_{2m-j}) & \text{if} \quad m < k < 2m-j.\end{cases}\\
S^k_j &\doteq \begin{cases} E^{2m-j}_k + E^k_{2m-j} & \text{if} \quad j < k \leq m;\\ -E^k_j - E^j_k & \text{if} \quad m < k < 2m-j.\end{cases}\\
T^k_j &\doteq \begin{cases} i(E^{2m-j}_k - E^k_{2m-j}) & \text{if} \quad j < k \leq m;\\ i(E^j_k - E^k_j) & \text{if} \quad m < k < 2m-j.\end{cases}
\end{aligned}
\tag{12}
$$

The proof of claims * and ** is a straightforward check, using the integrability conditions (2). One has to check only the case $j = 1, \quad 2 \leq k \leq 2m-2$. Indeed, let $\mathfrak{g}$ be either $su(m, m-1)$ or $sl(2m-1, \mathbb{R})$. The operators P, J, preserve the subspace

$$
span\{U^1, V^1, S^1, T^1, U^k_1, V^k_1, S^k_1, T^k_1 : 1 < k < 2m-1\}. \tag{13}
$$

Thus if we remove the external two rows and columns (13) in all matrices of $su(m, m-1)$ $(sl(2m-1, \mathbb{R}))$, then we get the Lie subalgebras

$$
su(m-1, m-2) \subset su(m, m-1), \quad (sl(2m-3, \mathbb{R}) \subset sl(2m-1, \mathbb{R})).
$$

By their definition, the operators P, J preserve these subalgebras and restricting to them we get the induction step. □

Remark 3.1. *In the last paragraph of the proof above, we have shown, that our complex product structures allow embeddings*

$$
\begin{gathered}
SU(2,1) \subset SU(3,2) \subset \cdots \subset SU(m, m-1);\\
SL(3) \subset SL(5) \subset \cdots \subset SL(2m-1)
\end{gathered}
$$

as complex product submanifolds. Similar induction was used by Joyce [8] *to get hypercomplex structures on compact reductive groups.*

Using the symmetry of (P, J) defined above, we can get some further examples

Remark 3.2. *Let $\mathfrak{g}$ be either $su(m, m-1)$ or $sl(2m-1, \mathbb{R})$. The operators (P, J) of the proof of Theorem 1.1 preserve the subspace*

$$
span\{E^j_m, E^m_k : 1 \leq j, k \leq 2m-1\} \cap \mathfrak{g}
$$

Thus, if we remove the middle row and column in all matrices of $su(m, m-1)$ (resp. $sl(2m-1, \mathbb{R})$), then we get the subalgebras

$$
u(m-1, m-1) \subset su(m, m-1) \quad (\text{ respectively } gl(2(m-1), \mathbb{R}) \subset sl(2m-1, \mathbb{R}))
$$

The action of (P, J) preserves these subalgebras also, so for each $m \geq 1$ we have found a complex product structure on $U(m, m)$ ($GL(2m, \mathbb{R})$) and embedded it as a complex product submanifold in $SU(m+1, m)$ (respectively $SL(2m+1, \mathbb{R})$).

We shall now introduce parameters in the definition of J, P. Let $\mathfrak{g}$ be one of the algebras $su(m, m-1)$ or $sl(2m-1, \mathbb{R})$. Using the notation of the proof of Theorem 1.1 we define an Abelian subalgebra

$$\mathfrak{z} \doteq \mathbb{R}U^1 \oplus \cdots \oplus \mathbb{R}U^{m-1} \subset \mathfrak{g}. \tag{14}$$

We have

Proposition 3.1. *Let $\mathfrak{g}$ be any of the Lie algebras $su(m, m-1), sl(2m-1, \mathbb{R})$. Let $Z^1, \ldots, Z^{m-1}$ be any base for the subalgebra $\mathfrak{z}$ defined in (14). Let*

$$V^j, S^j, T^j, U_j^k, V_j^k, S_j^k, T_j^k,$$

be as in the proof of Theorem 1.1. Then the operators $J, P : \mathfrak{g} \longmapsto \mathfrak{g}$ defined by

$$\begin{aligned} & P^2 = -J^2 = 1 \\ & J(Z^j) \doteq V^j; \quad J(S^j) \doteq T^j; \qquad P(Z^j) \doteq T^j, \quad P(V^j) \doteq S^j; \\ & J(U_j^k) \doteq V_j^k; \quad J(S_j^k) \doteq T_j^k; \qquad P(U_j^k) \doteq T_j^k; \quad P(V_j^k) \doteq S_j^k. \end{aligned} \tag{15}$$

give a complex product structure on $\mathfrak{g}$.

Proof. We have to prove integrability. Again a straightforward check of the Nijenhuis conditions gives the result. It helps to check first that

$$[Z, J(X)] = J([Z, X]); \quad [Z, P(X)] = P([Z, X]), \qquad Z \in \mathfrak{z}, X \in \mathfrak{g}.$$

□

Now we turn to metric properties of the above complex product structures.

Proof. of Theorem 1.4

The scalar product

$$\langle X, Y \rangle \doteq \frac{1}{2} tr(XY), \quad X, Y \in su(m, m-1), \tag{16}$$

is proportional to the Killing form.

First we treat $su(m, m-1)$. The product (16) is negative definite on the subalgebra $\mathfrak{z}$ (defined in (14)). Let $Z^1, \ldots, Z^{m-1}$ be any orthonormal base of $\mathfrak{z}$ with respect to $\langle ., . \rangle$. We take $V^j, S^j, T^j, U_j^k, V_j^k, S_j^k, T_j^k$, as defined in formulae (11), (12). Then the base

$$Z^j, V^j, S^j, T^j, U_j^k, V_j^k, S_j^k, T_j^k, \tag{17}$$

is orthogonal with respect to $\langle ., . \rangle$ and

$$\begin{aligned} \|Z^j\|^2 = \|V^j\|^2 = \|U_j^k\|^2 = \|V_j^k\|^2 = -1; \\ \|S^j\|^2 = \|T^j\|^2 = \|S_j^k\|^2 = \|T_j^k\|^2 = 1. \end{aligned}$$

It is then obvious from the definition of P, J (in formula (15)), that $\langle ., . \rangle$ is compatible, thus $su(m, m-1)$ is settled.

We present $sl(2m-1, \mathbb{C}) = su(m, m-1) \oplus isu(m, m-1)$. Then we keep the action of (P, J) on $su(m, m-1)$ as in Proposition 3.1 with respect to the base (17), and extend them to $sl(2m-1, \mathbb{C})$ by

$$\begin{aligned} & J(iZ^j) \doteq iV^j; \quad J(iS^j) \doteq iT^j; \qquad P(iZ^j) \doteq iT^j, \quad P(iV^j) \doteq iS^j; \\ & J(iU_j^k) \doteq iV_j^k; \quad J(iS_j^k) \doteq iT_j^k; \qquad P(iU_j^k) \doteq iT_j^k; \quad P(iV_j^k) \doteq iS_j^k. \end{aligned} \tag{18}$$

The scalar product $\langle X, Y \rangle$ defined in (16) (now we apply it to $X, Y \in sl(2m-1, \mathbb{C})$) is proportional to the Killing form on $sl(2m-1, \mathbb{C})$. Then

$$K(X, Y) \doteq \Re \langle X, Y \rangle$$

is proportional to the Killing form on $sl(2m-1,\mathbb{C})^{\mathbb{R}}$ (see e.g. [5], Ch. III, Lemma 6.1). Thus the set of

$$Z^j, V^j, S^j, T^j, U^k_j, V^k_j, S^k_j, T^k_j,$$
$$iZ^j, iV^j, iS^j, iT^j, iU^k_j, iV^k_j, iS^k_j, iT^k_j.$$

with any indices satisfying (7), is an orthogonal base of $sl(2m-1,\mathbb{C})^{\mathbb{R}}$ and

$$K(iZ,iZ) = K(iU,iU) = K(iV,iV) = K(S,S) = K(T,T) = 1,$$
$$K(Z,Z) = K(U,U) = K(V,V) = K(iS,iS) = K(iT,iT) = -1.$$

Whence the extention (18) of (P,J) to $sl(2m-1,\mathbb{C})^{\mathbb{R}}$ is compatible with the Killing metric. $\square$

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